New weighted integral inequalities and fractional consequences

JUAN EDUARDO Nápoles Valdés
UNNE-FaCENA, UTN-FRRE, Argentina
jnapoles@exa.unne.edu.ar

In Mathematics, the notion of convex function plays a very prominent role due to its multiple applications and its theoretical overlaps with various other areas of science (see [10] for more information).

One of the most important inequalities for convex functions is the well-known Hermite-Hadamard inequality (see [4,5] and [9] for additional details):

$\psi\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\psi(x)dx \leq \frac{\psi(\nu_1)+\psi(\nu_2)}{2}$.

In the last 25 years, we have witnessed a great growth in the number of researchers and their productions, interested in the Hermite-Hadamard Inequality. These productions have focused on the following work directions:

1) Using different notions of convexity.

2) Refinement of the mesh used (there is a crucial issue in this direction of work, suppose we use instead of a and b, the ends of the interval, the points a, $\frac{a+b}{2}$ and b, then we must ensure that at the midpoint, the integral operator used, does not have a jump, since the result would not be guaranteed in all $[a,b]$).

3) Improved estimates of the left and right members of Hermite-Hadamard inequality.

4) Using new generalized and fractional integral operators.

In [2] we presented the following definitions.

Let $h:[0,1]\to\mathbb{R}$ be a nonnegative function, $h\neq 0$ and $\psi: I=[0,+\infty)\to[0,+\infty)$. If inequality

$\psi\left(\tau\xi+m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi)+m(1-h^s(\tau))\psi\left(\varsigma\right)$

is fulfilled for all $\xi,\varsigma\in I$ and $\tau\in[0,1]$, where $m\in[0,1]$, $s\in[-1,1]$. Then a function ψ is called a (h,m)-convex modified of the first type on I.

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is fulfilled for all $\xi,\varsigma\in I$ and $\tau\in[0,1]$, where $m\in[0,1]$, $s\in[-1,1]$. Then a function ψ is called a (h,m)-convex modified of the second type on I.

Interested readers can verify that the previous definitions contain many of the known notions of convexity.

A new way to define an integral operator, and take a first step in generalizing the known results, is to consider a certain weight in the definition of the operator integral, as follows:

(see [2]) Let $\phi\in L_1[a_1,a_2]$ and let w be a continuous and positive function, $w: I\to\mathbb{R}^+$, with first derivative integrables on I°. Then the weighted fractional integrals are defined by (right and left respectively):

$I_{a_1+}^{w}\phi(t)=\int_{a_1}^{t}w'''\left(\frac{a_2-t}{a_2-a_1}\right)\phi(t)\,dt,\quad t>a_1$

$I_{a_2-}^{w}\phi(t)=\int_{t}^{a_2}w'''\left(\frac{t-a_1}{a_2-a_1}\right)\phi(t)\,dt,\quad t<a_2.$

The consideration of the third derivative of the weight function w is given by the nature of the problem to be solved, it can also be considered the first and second derivative.

To have a clearer idea of the amplitude of the previous Definition, let's consider some particular cases of the weight w''':

a) Putting $w'''(t)\equiv 1$, we obtain the classical Riemann integral.

b) If $w'''(t)=\frac{t^{(\alpha-1)}}{\Gamma(\alpha)}$, then we obtain the Riemann-Liouville fractional integral.

c) With convenient weight choices w''' we can get the k-Riemann-Liouville fractional integral right and left, the right-sided fractional integrals of a function ψ with respect to another function h on $[a,b]$ (see [1]), the right and left integral operator of [6], the right and left sided generalized fractional integral operators and the integral operators of [7] and [8], can also be obtained from above Definition by imposing similar conditions to w'.

d) Of course there are other known integral operators, fractional or not, that can be obtained as particular cases of the previous one, but we leave it to interested readers.

In 2015, Caputo and Fabrizio proposed the following operator (see [3]:

Let $0 < \alpha \leq 1$, $f \in AC^1[\nu_1, \nu_2]$. The right-sided and left-sided Caputo-Fabrizio fractional derivative of order α are defined as follows:

$\left({}^C D^\alpha_{\nu_1+} f\right)(t) = \frac{B(\alpha)}{1-\alpha}\int_{\nu_1}^t f'(x) e^{-\frac{\alpha(t-x)^\alpha}{1-\alpha}} dx, t > \nu_1$

$\left({}^C D^\alpha_{\nu_2-} f\right)(t) = -\frac{B(\alpha)}{1-\alpha}\int_t^{\nu_2} f'(x) e^{-\frac{\alpha(x-t)^\alpha}{1-\alpha}} dx, t < \nu_2,$

where $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$.

Their corresponding integral operators given by:

Let $0 < \alpha \leq 1$, $f \in AC^1[\nu_1, \nu_2]$. The right-sided and left-sided Caputo-Fabrizio integral of order α are defined as follows:

$\left({}^{CF} I^\alpha_{\nu_1+} f\right)(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)}\int_{\nu_1}^t f(y)dy, t > \nu_1$

$\left({}^{CF} I^\alpha_{\nu_2-} f\right)(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)}\int_t^{\nu_2} f(y)dy, t < \nu_2,$

where $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$.

In this paper we obtain new integral inequalities, within the framework of (h, m)-convex functions modified of second type, using weighted integrals. Various consequences for fractional integrals of type CF are presented throughout the work.

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